Solid Modeling

CS418 Computer Graphics

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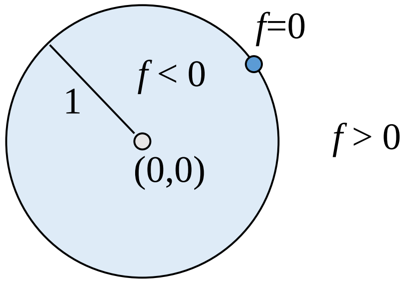
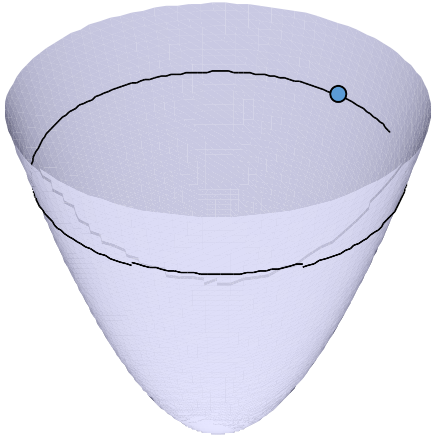
Implicit Surfaces

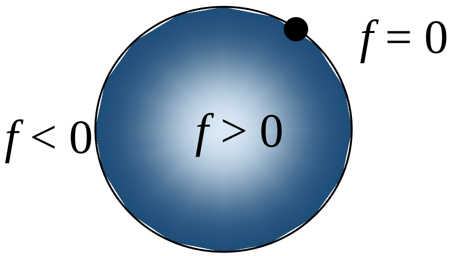
- Real function $f\,(x,y,z)$
- Classifies points in space
- Image synthesis (sometimes)
  - inside $f > 0$
  - outside $f < 0$
  - on the surface $f = 0$
- CAGD: inside $f < 0$, outside $f > 0$

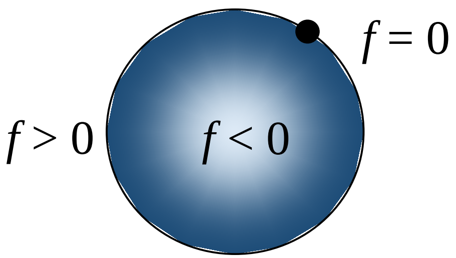


Circle example
$f(x,y) = x^2 + y^2 - 1$



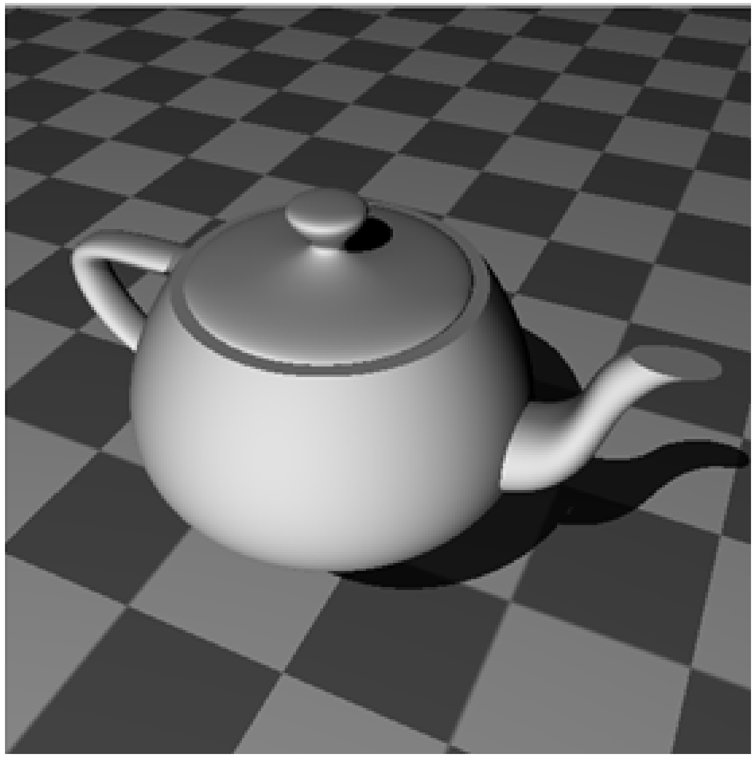
$z = f(x,y)$

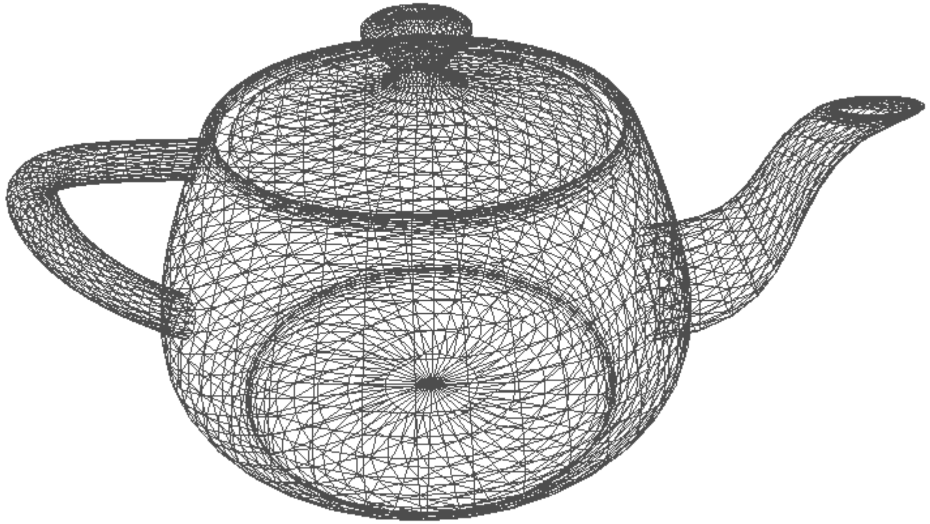




Why Use Implicits?

- v. polygons
  - smoother
  - compact, fewer higher-level primitives
  - harder to display in real time
- v. parametric patches
  - easier to blend
  - no topology problems
  - lower degree
  - harder to parameterize
  - easier to ray trace
  - well defined interior





Surface Normals

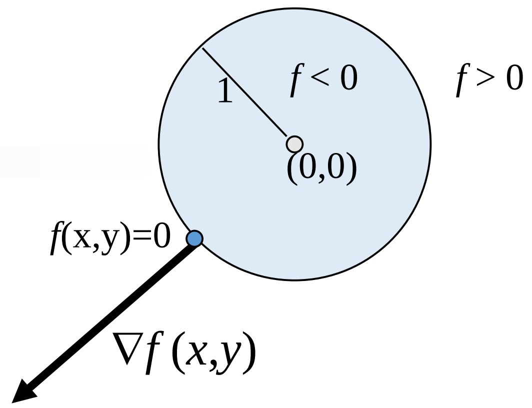
- Surface normal is parallel to the function gradient

  $$\nabla f(x,y,z) = (\delta f/\delta x,\ \delta f/\delta y,\ \delta f/\delta z)$$

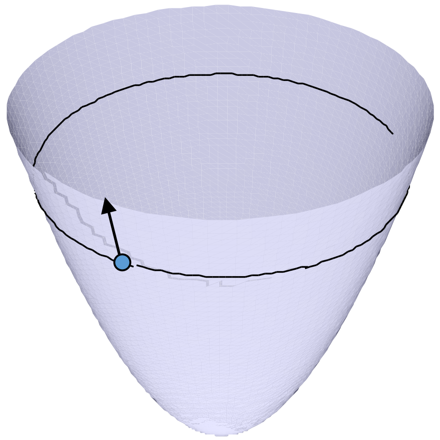
- Gradient not necessarily unit length

  $$\mathbf{n} = \nabla f\,(x,y,z)/\|\nabla f\,(x,y,z)\|$$

- Gradient points in direction of increasing f
  - Outward when $f < 0$ denotes interior
  - Inward when $f > 0$ denotes interior



Circle example
$f\,(x,y) = x^2 + y^2 - 1$
$\nabla f\,(x,y) = (2x,\ 2y)$



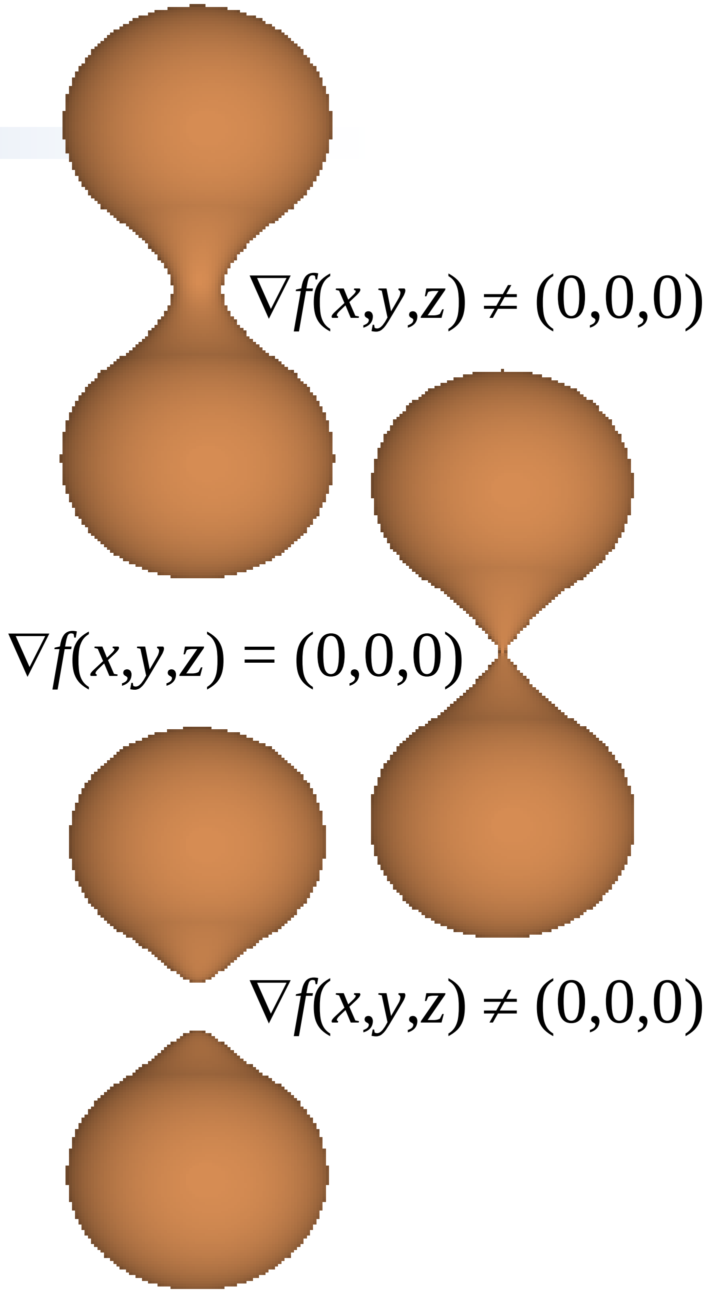
$z = f(x,y)$

Smoothness

- Surface $f^{-1}(0)$ is a smooth "manifold" if zero is a "regular" value of f

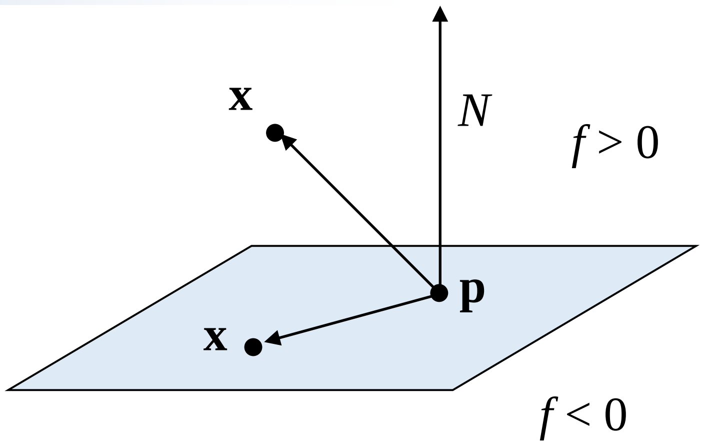
- Surface is "manifold" if the infinitesimal neighborhood around any point can be deformed into a simple flat region

- Zero is a "regular" value means that at any point (x,y,z) where f(x,y,z) = 0, then $\nabla f(x,y,z) \neq (0,0,0)$



Plane

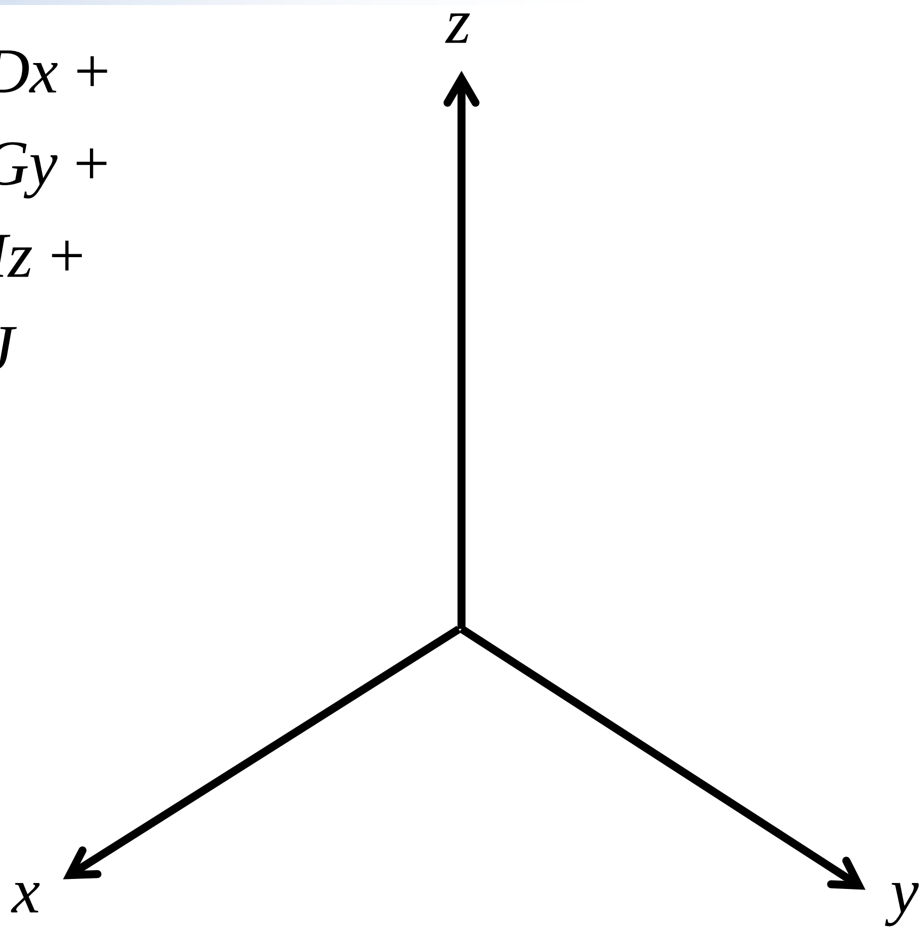
- Plane bounds half-space
- Specify plane with point **p** and normal N
- Points in plane **x** are perp. to normal N
- f is distance if $\|N\| = 1$



$$\boxed{f(\mathbf{x}) = (\mathbf{x}-\mathbf{p})\cdot N}$$

Quadrics

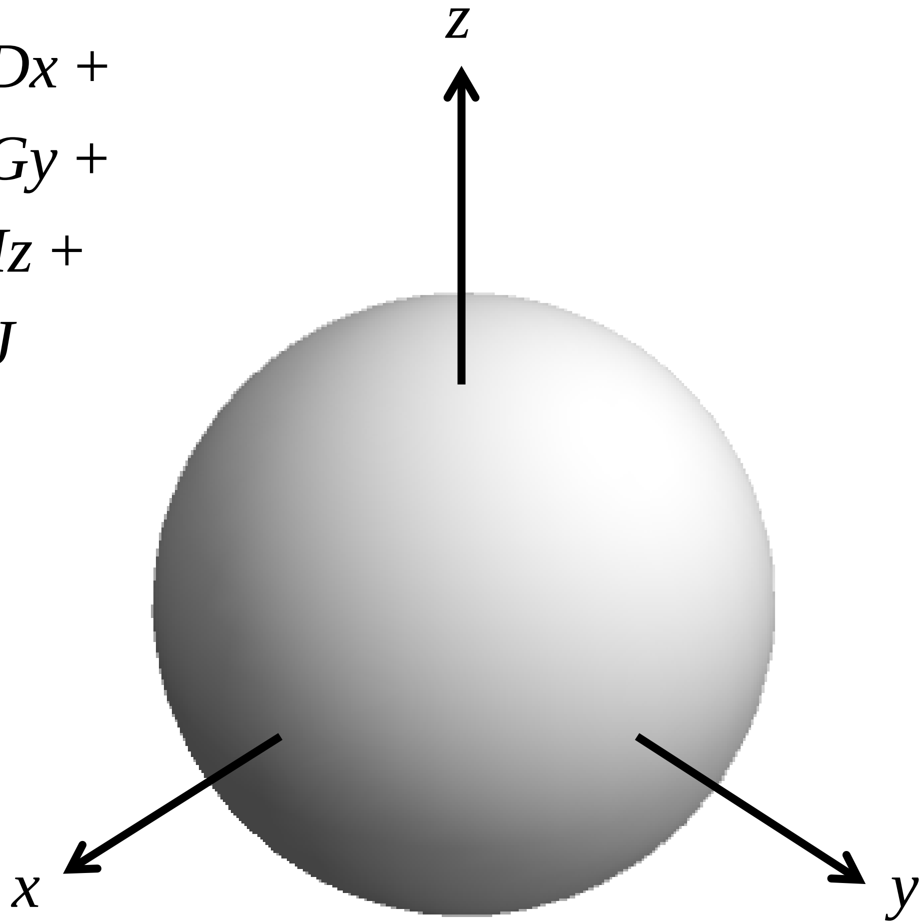
$$
\begin{aligned}
f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx &+ \\
Ey^2 + 2Fyz + 2Gy &+ \\
Hz^2 + 2Iz &+ \\
J&
\end{aligned}
$$



Quadrics

$$f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J$$

- Sphere: $x^2 + y^2 + z^2 - 1 = 0$



Quadrics

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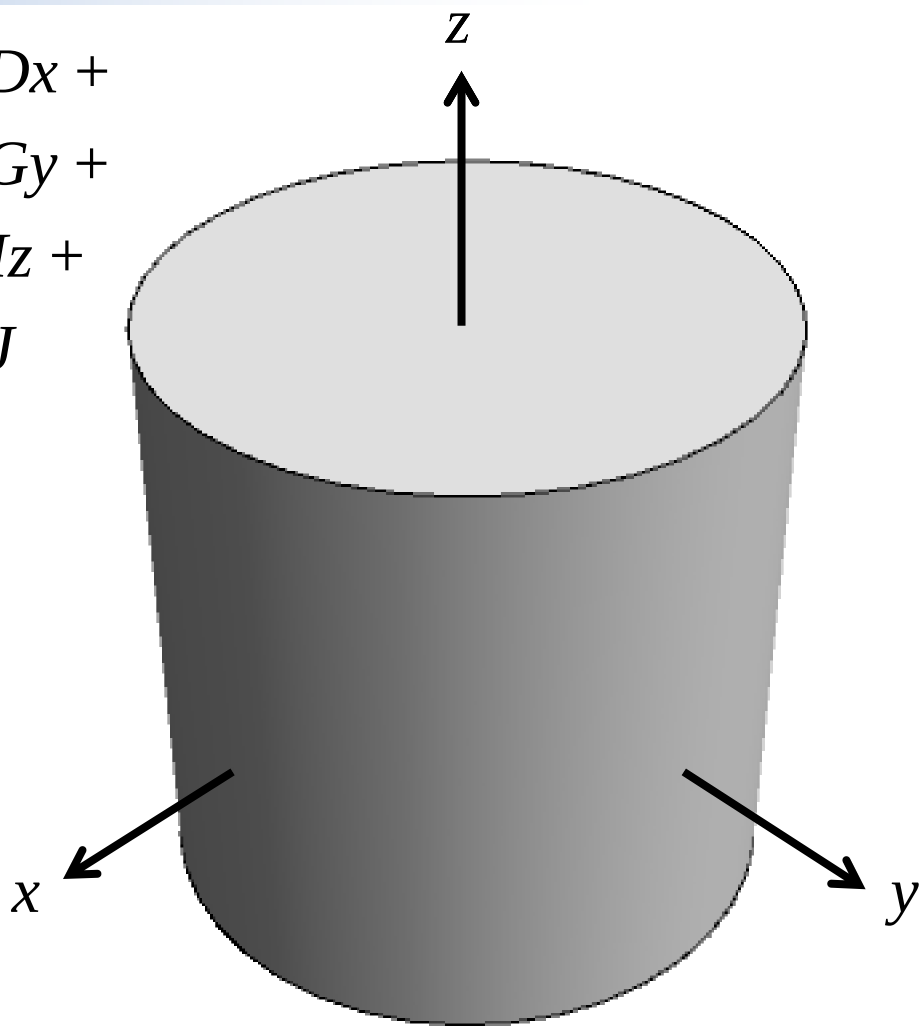
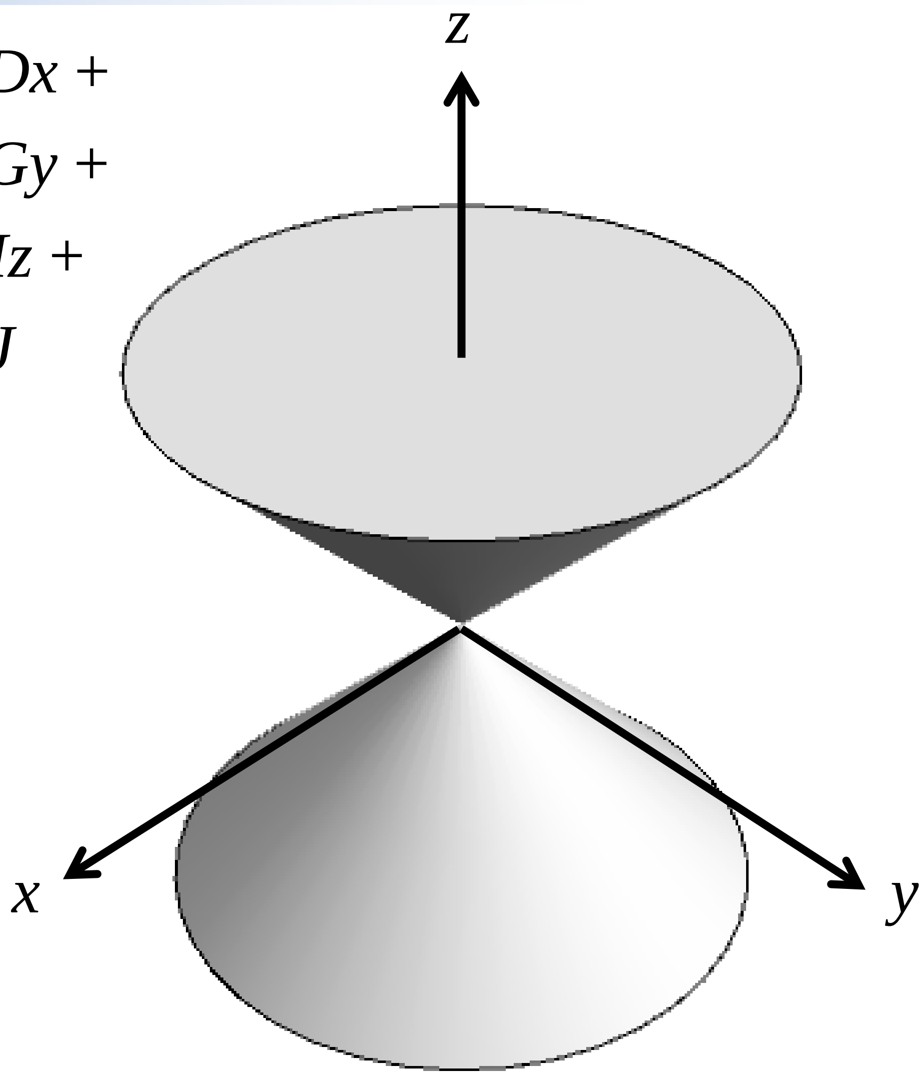
- Sphere: $x^2 + y^2 + z^2 - 1 = 0$
- Cylinder: $x^2 + y^2 - 1 = 0$



Quadrics

$$f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J$$

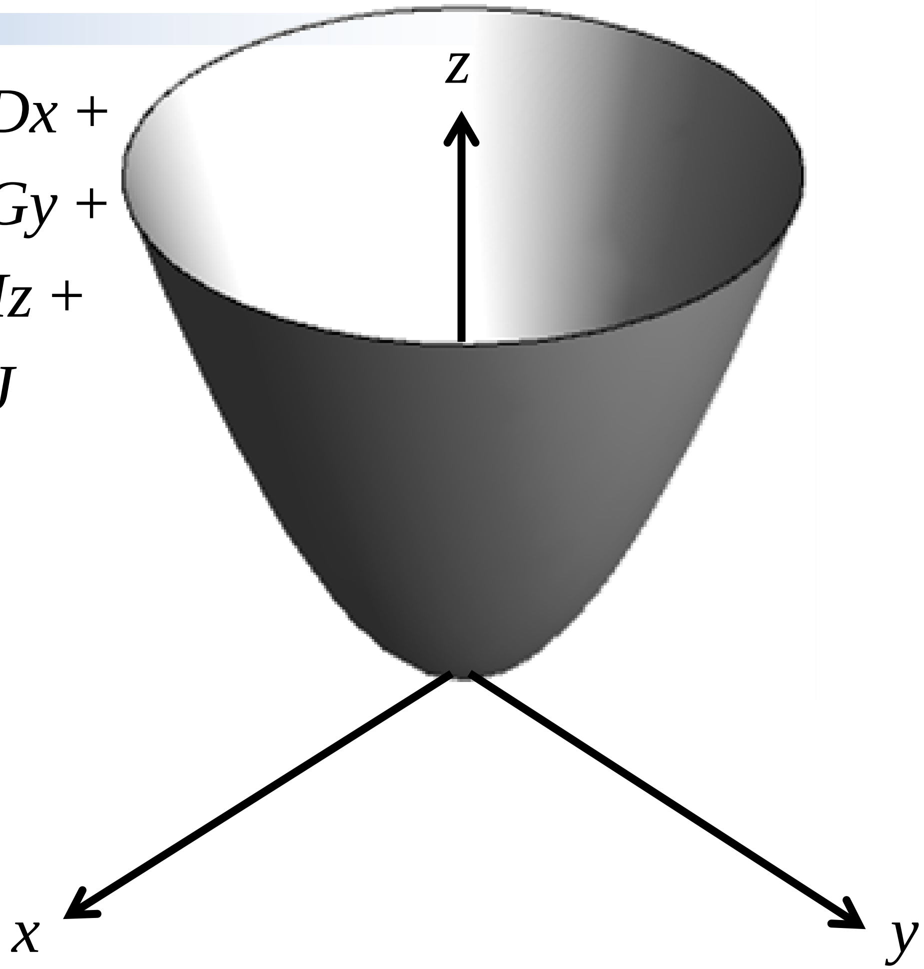
- Sphere: $x^2 + y^2 + z^2 - 1 = 0$
- Cylinder: $x^2 + y^2 - 1 = 0$
- Cone: $x^2 + y^2 - z^2 = 0$



Quadrics

$$f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J$$

- Sphere: $x^2 + y^2 + z^2 - 1 = 0$
- Cylinder: $x^2 + y^2 - 1 = 0$
- Cone: $x^2 + y^2 - z^2 = 0$
- Paraboloid: $x^2 + y^2 - z = 0$



Homogeneous Quadrics

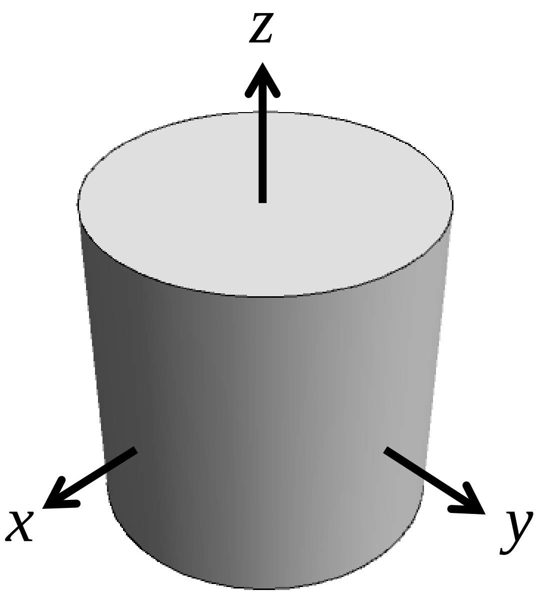
$$f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J$$

$$f(x,y,z) = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$f(\mathbf{x}) = \mathbf{x}^T Q\, \mathbf{x}$$

Transforming Quadrics

- Given a quadric Q with implicit surface $\mathbf{x}^{\mathrm{T}} Q\, \mathbf{x} = 0$
- What is the matrix Q' of the quadric transformed by M



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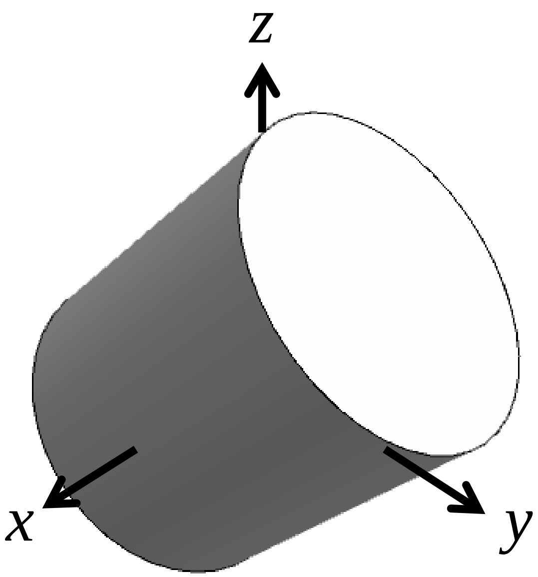

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Transforming Quadrics

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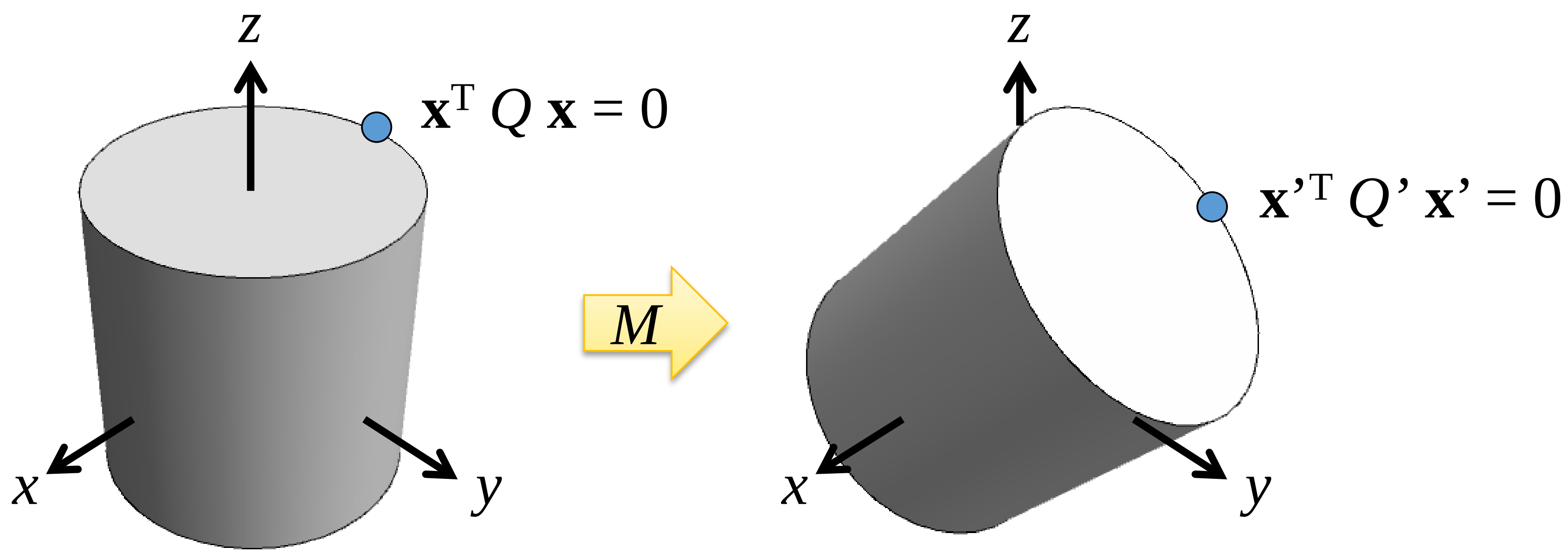
- Let $\mathbf{x}' = M\, \mathbf{x}$
- Find Q' such that $\mathbf{x}'^T Q'\, \mathbf{x}' = 0$



Transforming Quadrics

- Given a quadric Q with implicit surface $\mathbf{x}^T Q \mathbf{x} = 0$
- What is the matrix Q' of the quadric transformed by M
- Let $\mathbf{x}' = M \mathbf{x}$
- Find Q' such that $\mathbf{x}'^T Q' \mathbf{x}' = 0$
- Since $\mathbf{x} = M^{-1} \mathbf{x}'$ we have

$$(M^{-1} \mathbf{x}')^T Q (M^{-1} \mathbf{x}') = 0$$

$$\mathbf{x}'^T (M^{-1})^T Q M^{-1} \mathbf{x}' = 0$$

- So $Q' = (M^{-1})^T Q M^{-1}$

Transforming Quadrics

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- So $Q' = (M^{-1})^T Q M^{-1}$

(Since $\mathbf{x}$ is homogeneous, and we're evaluating $\mathbf{x}^T Q \mathbf{x} = 0$ we don't care about scale, so we can use the easier-to-compute adjoint M* instead of the inverse M^{-1}.)

Torus

- Product of two implicit circles

  $(x - R)^2 + z^2 - r^2 = 0$

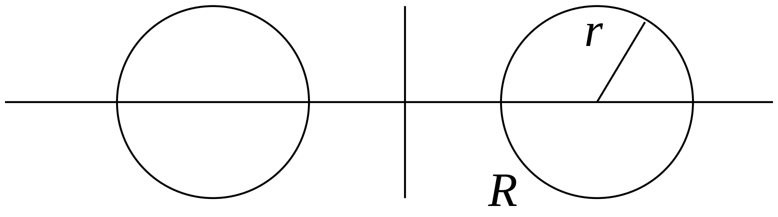
  $(x + R)^2 + z^2 - r^2 = 0$

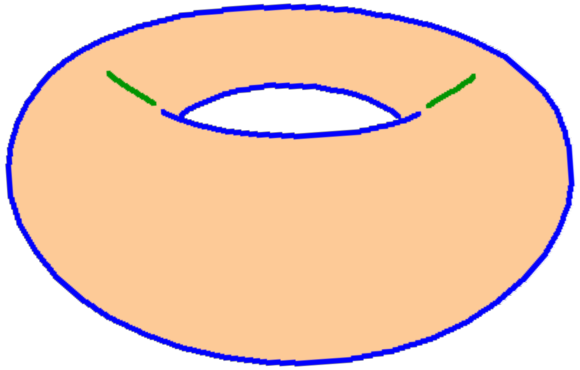
  $((x - R)^2 + z^2 - r^2)((x + R)^2 + z^2 - r^2)$

  $(x^2 - Rx + R^2 + z^2 - r^2)\ (x^2 + Rx + R^2 + z^2 - r^2)$

  $x^4 + 2x^2z^2 + z^4 - 2x^2r^2 - 2z^2r^2 + r^4 + 2x^2R^2 + 2z^2R^2 - 2r^2R^2 + R^4$

  $(x^2 + z^2 - r^2 - R^2)^2 + 4z^2R^2 - 4r^2R^2$





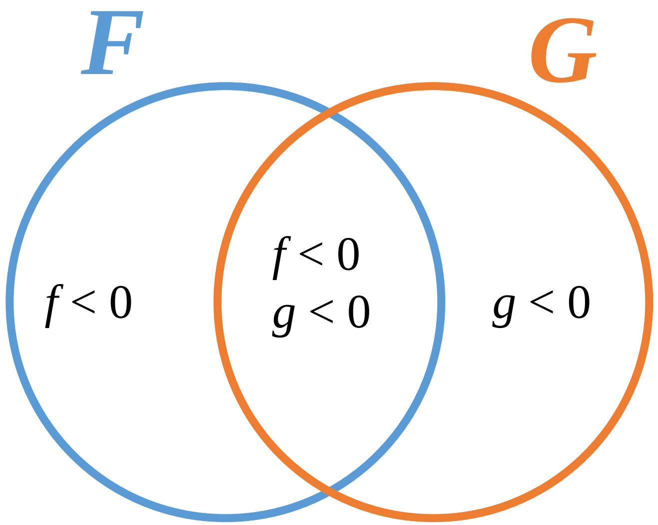
- Surface of rotation

  replace x^2 with $x^2 + y^2$

  $f(x,y,z) = (x^2 + y^2 + z^2 - r^2 - R^2)^2 + 4R^2(z^2 - r^2)$

Constructive Solid Geometry

- Let shape F be implicitly defined by $f(\mathbf{x})$
- Let shape G be implicitly defined by $g(\mathbf{x})$



Constructive Solid Geometry

- Let shape F be implicitly defined by $f(\mathbf{x})$
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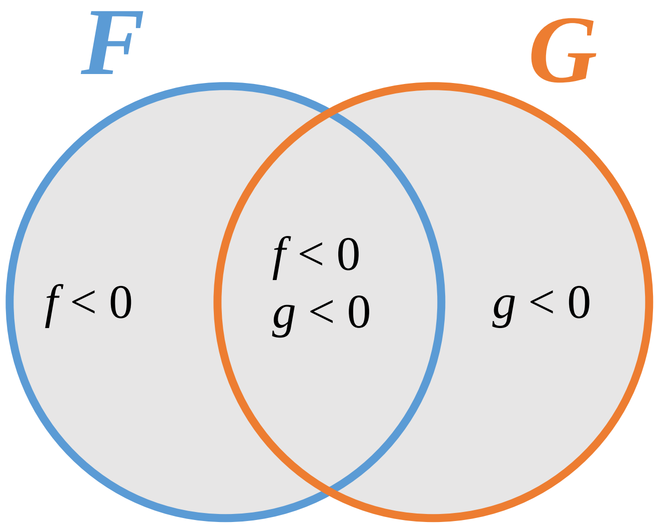
- The union $H = F \cup G$ is defined by

$$h(\mathbf{x}) = \min f(\mathbf{x}), g(\mathbf{x})$$



$$\boldsymbol{H} = F \cup G$$

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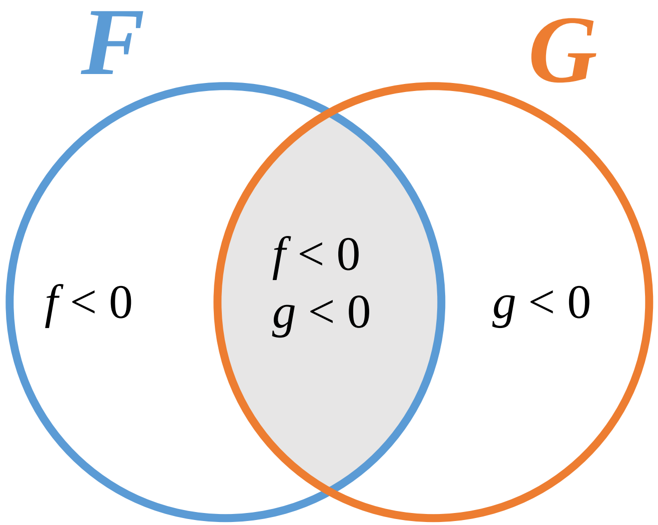

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- The intersection $H = F \cap G$ is defined by

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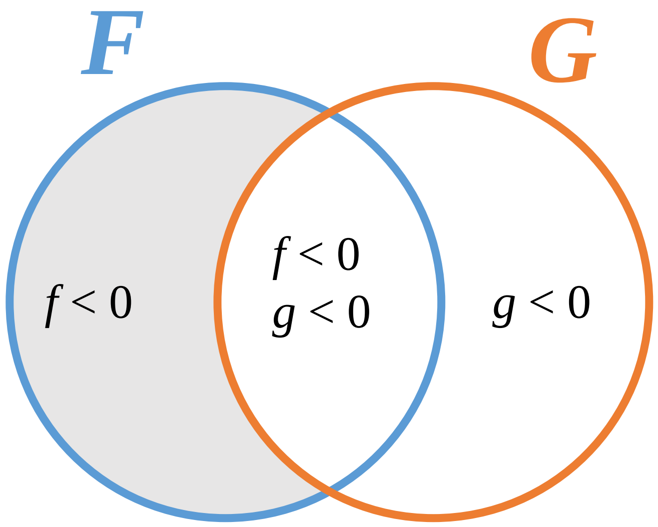
- The difference $H = F - G$ is defined by

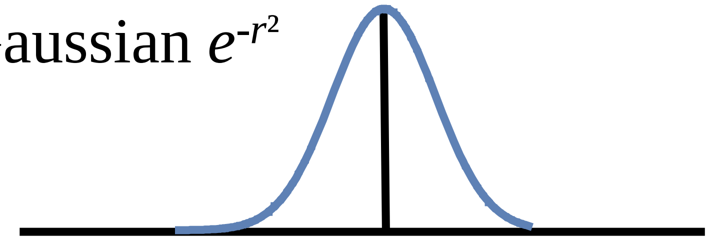
$$h(\mathbf{x}) = \max f(\mathbf{x}), -g(\mathbf{x})$$



$$\boldsymbol{H} = F - G$$

Blobs

- Gaussian e^{-r^2}

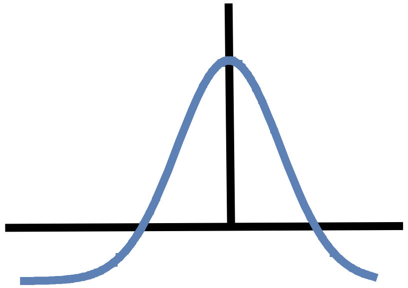


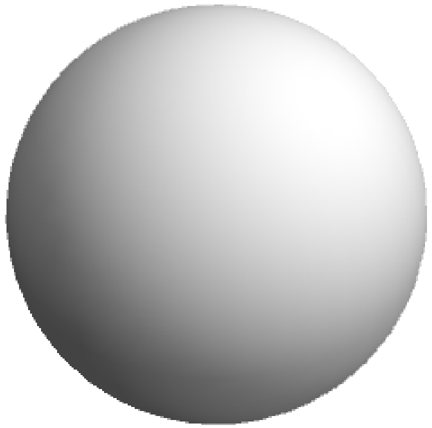
- Radius function

$$r^2(\mathbf{x}) = (\mathbf{x} - \mathbf{c})\cdot(\mathbf{x} - \mathbf{c})$$

- Gaussian sphere

$$f(\mathbf{x}) = -T + e^{-r^2(\mathbf{x})}$$



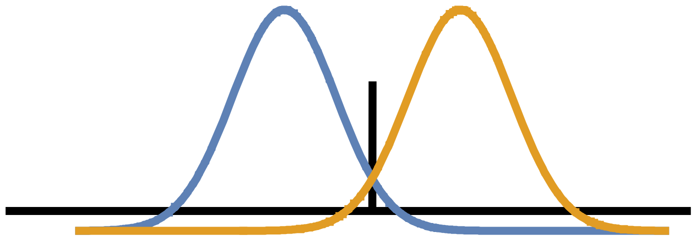


- For this formulation of implicit surfaces, function is *positive* inside the object

Blobs

- Union of spheres

$$f(\mathbf{x}) = -T + \min\, e^{-r_1^2(\mathbf{x})},\ e^{-r_2^2(\mathbf{x})}$$



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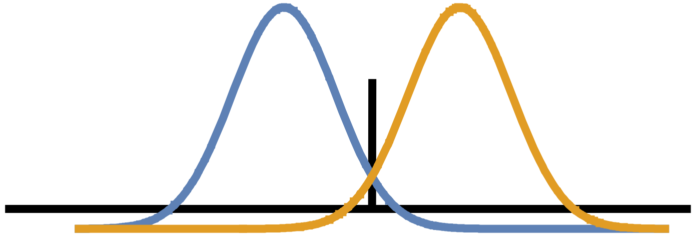

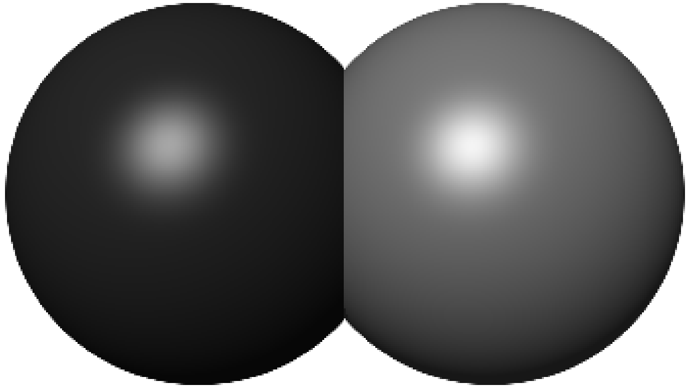

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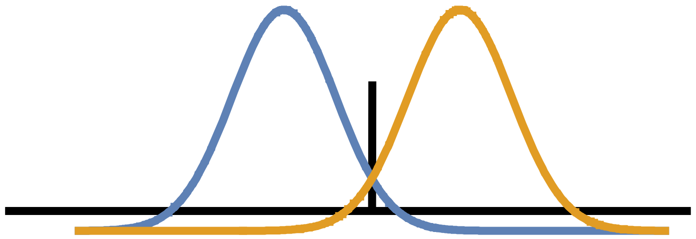
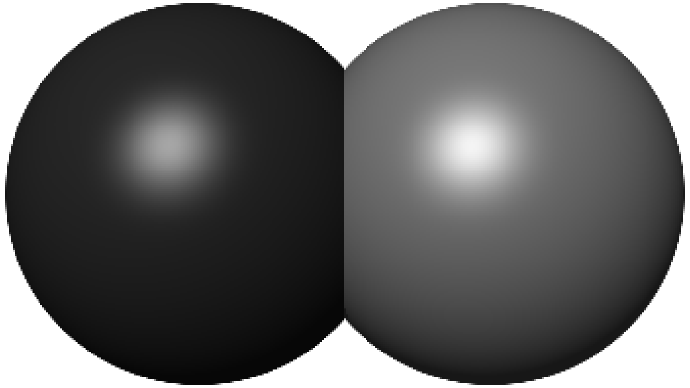
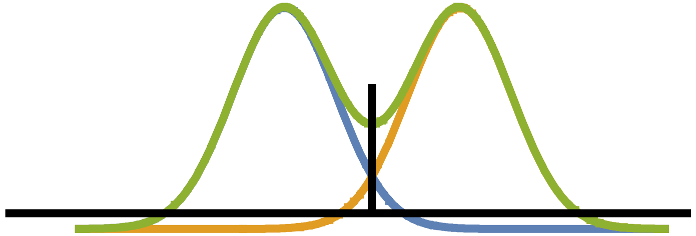




- Blended union of spheres

$$f(\mathbf{x}) = -T + e^{-r_1^2(\mathbf{x})} + e^{-r_2^2(\mathbf{x})}$$



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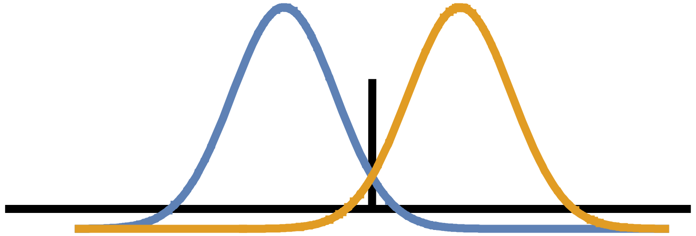

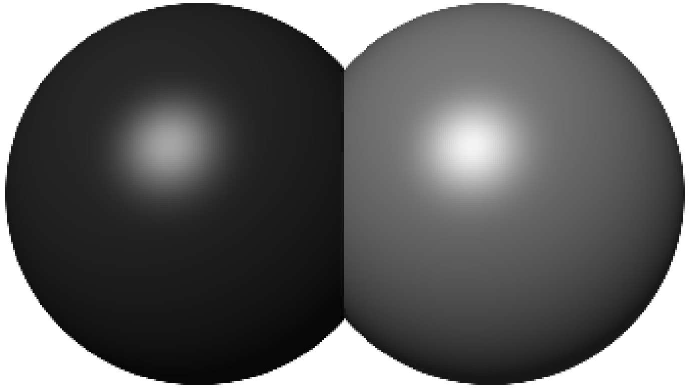

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